

A Note on Rotation Matrices

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Rotation matrices $d_{m,m'}^{(j)}(\theta)$ find wide applications in molecular and nuclear theory. They are known since the famous work of WIGNER¹ on the rotation group. As the derivation is complicated and needs group theory, a simple algebraic method will be valuable. The present note gives a simplification of Schwinger's solution of $d_{m,m'}^{(j)}(\theta)$ avoiding the unnecessary complicated generating function trick². We follow the customary notation and repeat for completeness the defining equations.

The problem is the evaluation of the matrixelement

$$d_{m,m'}^{(j)}(\theta) = \langle j m | \exp(-i \theta \hat{J}_y) | j m' \rangle. \quad (1)$$

SCHWINGER² introduced a boson representation of angular momentum in terms of the twodimensional isotropic harmonic oscillator creation and annihilation operators $\hat{a}_\pm^\pm$ and $\hat{a}_\pm$:

$$\hat{J}_+ = \hat{J}_x + i \hat{J}_y = \hat{a}_+^\pm \hat{a}_-, \quad (2)$$

$$\hat{J}_- = \hat{J}_x - i \hat{J}_y = \hat{a}_-^\pm \hat{a}_+, \quad (3)$$

$$\hat{J}_z = \frac{1}{2} (\hat{a}_+^\pm \hat{a}_+ - \hat{a}_-^\pm \hat{a}_-), \quad (4)$$

and

$$|j m'\rangle = [(j+m')! / (j-m')!]^{-1/2} (\hat{a}_+^\pm)^{j+m'} (\hat{a}_-^\pm)^{j-m'} |00\rangle. \quad (5)$$

By substitution, $\hat{J}_y$ is

$$\hat{J}_y = \frac{1}{2i} (\hat{a}_+^\pm \hat{a}_- - \hat{a}_-^\pm \hat{a}_+). \quad (6)$$

If the identity operator $\hat{I}$

$$\hat{I} = \exp(i \theta \hat{J}_y) \exp(-i \theta \hat{J}_y) \quad (7)$$

is introduced the problem of calculating $d_{m,m'}^{(j)}(\theta)$ is reduced to a canonical transformation

$$\begin{aligned} d_{m,m'}^{(j)}(\theta) &= [(j+m')! / (j-m')!]^{-1/2} \\ &\langle j m | \exp(-i \theta \hat{J}_y) (\hat{a}_+^\pm)^{j+m'} (\hat{a}_-^\pm)^{j-m'} \\ &\exp(i \theta \hat{J}_y) \exp(-i \theta \hat{J}_y) | 00 \rangle \end{aligned} \quad (8)$$

which will be performed by means of the well known expansion theorem:

$$\begin{aligned} \exp(\hat{B}) \hat{A}^k \exp(-\hat{B}) &= [\exp(\hat{B}) \hat{A} \exp(-\hat{B})]^k \\ &= \left(\hat{A} + [\hat{B}, \hat{A}] - \frac{1}{2!} [\hat{B}, [\hat{B}, \hat{A}]] + \dots \right)^k \\ &= \left(\sum_{s=0}^{\infty} \frac{1}{s!} \{\hat{B}^s, \hat{A}\}_- \right)^k. \end{aligned} \quad (9)$$

In the present case, the $\hat{B}$ operator is realized by $(-i \theta \hat{J}_y)$ and the $\hat{A}$ operator by $\hat{a}_\pm^\pm$ and $\hat{a}_\pm$, respectively. The commutator $[\hat{B}, \hat{A}]_-$ is $(\theta/2) \hat{a}_\pm^\pm$ and $-(\theta/2) \hat{a}_\pm$ in this special case. Therefore one obtains

$$\exp(-i \theta \hat{J}_y) \hat{a}_\pm^\pm \quad (10)$$

$$\exp(i \theta \hat{J}_y) = \hat{a}_\pm^\pm \cos(\theta/2) + \hat{a}_\pm \sin(\theta/2),$$

$$\exp(-i \theta \hat{J}_y) \hat{a}_\pm^\pm \quad (11)$$

$$\exp(i \theta \hat{J}_y) = \hat{a}_\pm^\pm \cos(\theta/2) - \hat{a}_\pm \sin(\theta/2).$$

As the operator $\exp(-i \theta \hat{J}_y)$ contains normalordered annihilation operators, all terms except the first one vanish in the series expansion. The final result for $d_{m,m'}^{(j)}(\theta)$ is written with the binomial theorem:

$$\begin{aligned} d_{m,m'}^{(j)}(\theta) &= [(j+m)! / (j-m)! (j+m')! / (j-m')!]^{-1/2} \\ &\sum_{k=0}^{j+m'} \sum_{l=0}^{j-m'} (-1)^l \binom{j+m'}{k} \binom{j-m'}{l} \cos^{2j-k-l}(\theta/2) \\ &\sin^{(k+l)}(\theta/2) \langle 00 | (\hat{a}_+^\pm)^{j+m} (\hat{a}_-^\pm)^{j-m} \\ &(\hat{a}_+^\pm)^{j+m'-k+l} (\hat{a}_-^\pm)^{j-m'-l+k} | 00 \rangle. \end{aligned} \quad (12)$$

The matrix element vanishes for all cases

$$\begin{aligned} j+m &\neq j+m'-k+l \\ \text{and/or} \quad j-m &\neq j-m'-l+k. \end{aligned} \quad (13)$$

If the expressions are equal, the matrixelement is $(j+m)! / (j-m)!$. By easy trigonometric manipulations the $d_{m,m'}^{(j)}(\theta)$ can be brought to the customary form of Jacobi-polynomials.

¹ E. P. WIGNER, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, New York 1959.

² J. SCHWINGER, On Angular Momentum, reprinted in: Quantum Theory of Angular Momentum, L. C. BIEDENHARN and H. VAN DAM, eds., Academic Press, New York 1965.

